Expositiones Mathematicae 28 (2010) 365–373

Contents lists available at ScienceDirect

Expositiones Mathematicae

journal homepage: www.elsevier.de/exmath

Isomorphisms and generalized derivations of some algebras ☆

Jiankui Li [a,*], Zhidong Pan [b], Jiren Zhou [a]

[a] Department of Mathematics, East China University of Science and Technology, Shanghai 200237, PR China
[b] Department of Mathematics, Saginaw Valley State University, University Center, MI 48710, USA

ARTICLE INFO

Article history:
Received 6 April 2009

2000 Mathematics Subject Classification:
primary, 47B47, 47L35

Keywords:
Ideal
Idempotent
Isomorphism
Jordan derivation

ABSTRACT

For a commutative subspace lattice $\mathcal{L}$ on a complex Hilbert space and a bounded bijective linear mapping h from $alg\,\mathcal{L}$ onto a unital Banach algebra $\mathcal{B}$, we show that if h satisfies $h(A)h(B)h(C)=0$ for all A,B,C in $alg\,\mathcal{L}$ with $AB=BC=0$ and $h(I)=I$, then h is an isomorphism. For a $\mathcal{J}$-subspace lattice $\mathcal{L}$ on a Banach space and the unital subalgebra $\mathcal{A}$ of $alg\,\mathcal{L}$ generated by finite-rank operators, we show that all generalized Jordan derivations from $\mathcal{A}$ to any unital $\mathcal{A}$-bimodule are generalized derivations.

1. Introduction

Let X be a complex Banach space and X^* be the topological dual of X. We denote by $B(X)$ the set of all bounded linear operators on X and by $F(X)$ the set of all finite-rank operators in $B(X)$. In this paper, a subspace of X is a closed linear manifold. By a *subspace lattice* on X, we mean a collection $\mathcal{L}$ of subspaces of X with 0 and X in $\mathcal{L}$ such that for every family $\{M_r\}$ of elements of $\mathcal{L}$, both $\cap M_r$ and $\vee M_r$ belong to $\mathcal{L}$, where $\vee M_r$ denotes the closed linear span of $\{M_r\}$. A totally ordered subspace lattice is called a *nest*. If $e \in X$ and $f \in X^*$, then the rank-one operator $x \mapsto f(x)e$ is denoted by $e \otimes f$. If $\mathcal{L}$ is a subspace lattice and $E \in \mathcal{L}$, we define

$$E_- = \vee\{F \in \mathcal{L} : E \nsubseteq F\} \quad \text{if } E \neq 0, E_+ = \cap\{F \in \mathcal{L} : F \nsubseteq E\} \quad \text{if } E \neq X.$$

☆ This work was completed with the support of NSF of China and the Ruth and Ted Braun Fellowship from the Saginaw Community Foundation.

* Corresponding author.
E-mail addresses: jiankuili@yahoo.com (J. Li), pan@svsu.edu (Z. Pan), zhoujiren1983@163.com (J. Zhou).

0723-0869/$ - see front matter © 2010 Elsevier GmbH. All rights reserved.
doi:10.1016/j.exmath.2010.01.001

For a subspace lattice $\mathcal{L}$ on X, $alg\,\mathcal{L}$ denotes the algebra of operators in $B(X)$ leaving each element of $\mathcal{L}$ invariant and $\mathcal{J}_{\mathcal{L}}$ denotes the subset of $\mathcal{L}$ defined by $\mathcal{J}_{\mathcal{L}} = \{L \in \mathcal{L} : L \neq (0) \text{ and } L_{-} \neq X\}$. A subspace lattice $\mathcal{L}$ on X is called a *$\mathcal{J}$-subspace lattice* if $L_{-} \cap L = 0$ for any $L \in \mathcal{J}_{\mathcal{L}}$, $X = \vee\{L : L \in \mathcal{J}_{\mathcal{L}}\}$ and $\cap\{L_{-} : L \in \mathcal{J}_{\mathcal{L}}\} = \{0\}$, see [19].

When X is a Hilbert space we change it to H. For a Hilbert space, we do not distinguish subspaces and the orthogonal projections onto them. A subspace lattice on a Hilbert H is called a *commutative subspace lattice* if it consists of mutually commuting projections.

Let $\mathcal{A}$ be an algebra and let $\mathcal{M}$ be an $\mathcal{A}$-bimodule. $\mathcal{A}$ is called *locally matric* if every finite subset of $\mathcal{A}$ is contained in a subalgebra of $\mathcal{A}$ which is isomorphic to $M_n(\mathbb{C})$. An additive mapping T from $\mathcal{A}$ into $\mathcal{M}$ is called a *left* (*right*) *multiplier* if $T(ab) = T(a)b$ ($T(ab) = aT(b)$) and T is called a *left* (*right*) *Jordan multiplier* if $T(x^2) = T(x)x$ ($T(x^2) = xT(x)$).

An additive (linear) mapping δ from $\mathcal{A}$ into $\mathcal{M}$ is called a *Jordan derivation* if $\delta(a^2) = \delta(a)a + a\delta(a)$ for every a in $\mathcal{A}$ and δ is called a *derivation* if $\delta(ab) = \delta(a)b + a\delta(b)$ for all a, b in $\mathcal{A}$. It is clear that every derivation is a Jordan derivation, but the converse is not true in general. We say that a Jordan derivation is non-trivial if it is not an ordinary derivation. An additive (linear) mapping δ from $\mathcal{A}$ into $\mathcal{M}$ is called a *generalized derivation* if there exists a derivation τ from $\mathcal{A}$ into $\mathcal{M}$ such that $\delta(ab) = \delta(a)b + a\tau(b)$ and δ is called a *generalized Jordan derivation* if there exists a Jordan derivation τ from $\mathcal{A}$ into $\mathcal{M}$ such that $\delta(a^2) = \delta(a)a + a\tau(a)$. An additive (linear) mapping d from $\mathcal{A}$ into $\mathcal{M}$ is called a *local Jordan derivation* if for any $a \in \mathcal{A}$ there exists a Jordan derivation δ_a from $\mathcal{A}$ into $\mathcal{M}$ such that $d(a) = \delta_a(a)$ and d is called a *local derivation* if for any $a \in \mathcal{A}$ there exists a derivation δ_a from $\mathcal{A}$ into $\mathcal{M}$ such that $d(a) = \delta_a(a)$. For more information on derivatives and Jordan derivatives, we refer to [1–4,11,12].

In Section 2, we study linear mappings h from an algebra $\mathcal{A}$ to an algebra $\mathcal{B}$ satisfying the following condition:

$$h(A)h(B)h(C) = 0 \quad \text{for all } A, B, C \in \mathcal{A} \quad \text{with } AB = BC = 0. \tag{$*$}$$

This condition is closely related local derivations. We show that for a commutative subspace lattice $\mathcal{L}$ on a Hilbert space, if a bounded bijective linear mapping h from $alg\,\mathcal{L}$ onto a unital Banach algebra $\mathcal{B}$ satisfies $(*)$ and $h(I) = I$ then h is an isomorphism.

In Section 3, we show that if $\mathcal{A}$ is the unital subalgebra of $alg\,\mathcal{L}$ generated by finite-rank operators, where $\mathcal{L}$ is a $\mathcal{J}$-subspace lattice on a Banach space, then all generalized Jordan derivations from $\mathcal{A}$ to a unital $\mathcal{A}$-bimodule are generalized derivations.

2. Isomorphisms

Throughout this section, we let $\mathcal{R}$ be the algebra generated by all idempotents in $\mathcal{A}$ and $\mathcal{I}$ be an ideal of $\mathcal{A}$ contained in $\mathcal{R}$.

Theorem 2.1. *Suppose $\mathcal{L}$ is a commutative subspace lattice and $\mathcal{A} = alg\,\mathcal{L}$. If h is a bounded bijective linear mapping from $\mathcal{A}$ onto a unital Banach algebra $\mathcal{B}$ satisfying $(*)$ and $h(I) = I$, then h is an isomorphism.*

For clarity, we break the proof of Theorem 2.1 into a few lemmas. We proceed by first gathering some equations to aid our proofs.

By [4, Theorem 3.1], if h is a linear mapping from an algebra $\mathcal{A}$ to an algebra $\mathcal{B}$ satisfying $(*)$ and $h(I) = I$ then

$$h(rxs) + h(r)h(x)h(s) = h(rx)h(s) + h(r)h(xs) \tag{2.1}$$

for $r, s \in \mathcal{R}$ and $x \in \mathcal{A}$.

It follows that h is a homomorphism on $\mathcal{R}$. In particular, if $e \in \mathcal{I}$ then $h(exs) = h(ex)h(s)$; this, combining with Eq. (2.1), implies

$$h(e)h(x)h(s) = h(e)h(xs) \tag{2.2}$$

for $e \in \mathcal{I}$, $s \in \mathcal{R}$ and $x \in \mathcal{A}$.

Similarly, we have $h(rxe)=h(r)h(xe)$; which, combining with Eq. (2.1), gives

$$h(r)h(x)h(e)=h(rx)h(e) \tag{2.3}$$

for $e\in\mathcal{I}$, $r\in\mathcal{R}$ and $x\in\mathcal{A}$.

For $e,f\in\mathcal{I}$ and $x\in\mathcal{A}$ we have $h(exf)=h(ex)h(f)=h(e)h(xf)$. Combining with Eq. (2.1), yields

$$h(exf)=h(e)h(x)h(f) \tag{2.4}$$

for $e,f\in\mathcal{I}$, and $x\in\mathcal{A}$.

For any $T\in alg\,\mathcal{L}$, define $\mathcal{A}(h,T)=\{A\in alg\,\mathcal{L}: h(TA)\text{-}h(T)h(A)=0\}$. Our goal is to show $\mathcal{A}(h,T)=alg\,\mathcal{L}$ for any $T\in alg\,\mathcal{L}$.

Lemma 2.2. *Let $\mathcal{L}$, $\mathcal{A}$ and h be as in Theorem* 2.1 *and let $\overline{\mathcal{R}}$ be the norm closure of $\mathcal{R}$. Then $h(AB)=h(A)h(B)$ for any $A,B\in\overline{\mathcal{R}}$.*

Proof. By Eq. (2.1), $h(EF)\text{-}h(E)h(F)=0$, for any idempotents $E,F\in\mathcal{A}$. The conclusion follows from the linearity and continuity of h. □

Similar to that of [16], we define $\mathcal{I}=span\{P(alg\,\mathcal{L})P^{\perp}:P\in\mathcal{L}\}$. It follows that $\mathcal{I}$ is an ideal of $alg\,\mathcal{L}$. Let Q be the projection onto the closure of the linear span of $\{PTP^{\perp}H:P\in\mathcal{L},T\in alg\,\mathcal{L}\}$. Then $Q\in\mathcal{L}$.

Lemma 2.3. *If $\mathcal{L}$ is a commutative subspace lattice then for any $A\in\mathcal{A}=alg\,\mathcal{L}$, $AQ^{\perp}\in\overline{\mathcal{R}}$.*

Proof. Since $PTP^{\perp}=P\text{-}(P\text{-}PTP^{\perp})$ and $P\text{-}PTP^{\perp}$ is an idempotent, every element in $\mathcal{I}$ is a linear combination of idempotents in $alg\,\mathcal{L}$. Write $AQ^{\perp}=QAQ^{\perp}+Q^{\perp}AQ^{\perp}$. We only need to show $Q^{\perp}AQ^{\perp}\in\overline{\mathcal{R}}$.

For any $P\in\mathcal{L}$, since $Q,A\in alg\,\mathcal{L}$, we get

$$Q^{\perp}AQ^{\perp}P=PQ^{\perp}AQ^{\perp}P. \tag{2.5}$$

On the other hand,

$$Q^{\perp}AQ^{\perp}P^{\perp}=Q^{\perp}PAP^{\perp}Q^{\perp}+Q^{\perp}P^{\perp}AQ^{\perp}P^{\perp}=0+Q^{\perp}P^{\perp}AQ^{\perp}P^{\perp}=P^{\perp}Q^{\perp}AQ^{\perp}P^{\perp}. \tag{2.6}$$

By (2.5) and (2.6), we have $Q^{\perp}AQ^{\perp}P=PQ^{\perp}AQ^{\perp}$; so $Q^{\perp}AQ^{\perp}\in\mathcal{L}'$, the commutant of $\mathcal{L}$. Note that a von Neumann algebra is the norm closure of the linear span of its projections. □

Lemma 2.4. *Let $\mathcal{L}$, $\mathcal{A}$ and h be as in Theorem* 2.1 *and let $\overline{\mathcal{R}}$ be the norm closure of $\mathcal{R}$. Then for any $E\in\mathcal{I}$ and $A\in\mathcal{A}$, $h(E)h(A)=0$ implies $EA=0$. Similarly, $h(A)h(E)=0$ implies $AE=0$.*

Proof. By Lemmas 2.2, 2.3, and Eq. (2.2), we have

$$h(EAQ^{\perp})=h(E)h(AQ^{\perp})=h(E)h(A)h(Q^{\perp})=0.$$

Thus $EAQ^{\perp}=0$.

By Eq. (2.4), for any $T\in\mathcal{A}$ and $P\in\mathcal{L}$,

$$h(EAPTP^{\perp})=h(E)h(A)h(PTP^{\perp})=0.$$

Thus $EAPTP^{\perp}=0$, which implies $EAQ=0$.

Combining $EAQ^{\perp}=0$ and $EAQ=0$, we get $EA=0$.

The proof of the other conclusion is similar. □

Lemma 2.5. *Let $\mathcal{L}$, $\mathcal{A}$ and h be as in Theorem* 2.1 *and let $\overline{\mathcal{R}}$ be the norm closure of $\mathcal{R}$. If $A\in\mathcal{A}$ satisfies $h(A)h(Q^{\perp})=0$, and $h(A)h(PTP^{\perp})=0$ for all $T\in\mathcal{A}$ and $P\in\mathcal{L}$, then $A=0$.*

Proof. By $h(A)h(PTP^{\perp})=0$ and Lemma 2.4, we have $APTP^{\perp}=0$. Thus $AQ=0$, so $A=AQ^{\perp}$. By Lemma 2.3, $h(A)h(Q)=h(AQ^{\perp})h(Q)=h(AQ^{\perp}Q)=0$.

Now $h(A)h(Q)=0$ and $h(A)h(Q^{\perp})=0$ imply $h(A)=0$. □

Lemma 2.6. *Let $\mathcal{L}$, $\mathcal{A}$ and h be as in Theorem* 2.1 *and let $\overline{\mathcal{R}}$ be the norm closure of $\mathcal{R}$. Then $h(EA)\text{-}h(E)h(A)=0$ and $h(AE)\text{-}h(A)h(E)=0$ for all $E\in\mathcal{I}$ and $A\in\mathcal{A}$.*

Proof. Take any $A \in \mathcal{A}$. We will show $A \in \mathcal{A}(h, E)$.

Write $A = AQ^{\perp} + AQ$. By Lemma 2.3, $AQ^{\perp} \in \mathcal{A}(h, E)$.

It remains to show $AQ \in \mathcal{A}(h, E)$.

Since h is surjective, let $B \in alg\,\mathcal{L}$ so that $h(B) = h(EAQ)\text{-}h(E)h(AQ)$.

Since $\mathcal{I}$ is an ideal of $\mathcal{A}$ generated by idempotents in $\mathcal{A}$, for any T in $\mathcal{A}$ and any $P \in \mathcal{L}$, $APTP^{\perp} \in \mathcal{I}$. By Eq. (2.4) and Lemma 2.3,

$$h(EAQ)h(PTP^{\perp})\text{-}h(E)h(AQ)h(PTP^{\perp}) = h(EAQPTP^{\perp})\text{-}h(EAQPTP^{\perp}) = 0.$$

It follows $h(B)h(PTP^{\perp}) = [h(EAQ)\text{-}h(E)h(AQ)]h(PTP^{\perp}) = 0$.

By Eq. (2.1), we have

$$h(B)h(Q^{\perp}) = [h(EAQ)\text{-}h(E)h(AQ)]h(Q^{\perp}) = [h(EA)h(Q)\text{-}h(E)h(A)h(Q)]h(Q^{\perp}) = 0.$$

Combining $h(B)h(PTP^{\perp}) = 0$ and $h(B)h(Q^{\perp}) = 0$, we get $B = 0$ by Lemma 2.5. $\square$

Lemma 2.7. *Let $\mathcal{L}$, $\mathcal{A}$ and h be as in Theorem 2.1 and let $\overline{\mathcal{R}}$ be the norm closure of $\mathcal{R}$. Then $h(CA)\text{-}h(C)h(A) = 0$ and $h(AC)\text{-}h(A)h(C) = 0$ for all $C \in \overline{\mathcal{R}}$ and $A \in \mathcal{A}$.*

Proof. First we show that for any $E \in \mathcal{R}$ and $A \in \mathcal{A}$, we have $A \in \mathcal{A}(h, E)$.

Write $A = AQ^{\perp} + AQ$. By Lemma 2.3, $AQ^{\perp} \in \mathcal{A}(h, E)$. We still need to show $AQ \in \mathcal{A}(h, E)$.

For any $T \in \mathcal{A}, P \in \mathcal{L}$, by Lemma 2.6,

$$h(EAQPTP^{\perp}) = h(EAQ)h(PTP^{\perp})$$

and

$$h(EAQPTP^{\perp}) = h(E)h(AQPTP^{\perp}) = h(E)h(AQ)h(PTP^{\perp}).$$

Thus $[h(EAQ)\text{-}h(E)h(AQ)]h(PTP^{\perp}) = 0$.

Since h is surjective, let $B \in alg\,\mathcal{L}$ so that $h(B) = h(EAQ)\text{-}h(E)h(AQ)$. It follows $h(B)h(PTP^{\perp}) = 0$. By Eq. (2.1), we have

$$h(B)h(Q^{\perp}) = [h(EAQ)\text{-}h(E)h(AQ)]h(Q^{\perp}) = [h(EA)h(Q)\text{-}h(E)h(A)h(Q)]h(Q^{\perp}) = 0.$$

Combining $h(B)h(PTP^{\perp}) = 0$ and $h(B)h(Q^{\perp}) = 0$, we get $B = 0$ by Lemma 2.5. The conclusion now follows from the continuity of h. $\square$

Proof of Theorem 2.1. For any $A, C, T \in \mathcal{A}$ and $P \in \mathcal{L}$, write $C = CQ^{\perp} + CQ$. By Lemmas 2.3 and 2.7, $CQ^{\perp} \in \mathcal{A}(h, A)$.

It remains to show $CQ \in \mathcal{A}(h, A)$.

By Lemma 2.6,

$$h(ACQPTP^{\perp}) = h(ACQ)h(PTP^{\perp})$$

and

$$h(ACQPTP^{\perp}) = h(A)h(CQPTP^{\perp}) = h(A)h(CQ)h(PTP^{\perp}).$$

Thus $[h(ACQ)\text{-}h(A)h(CQ)]h(PTP^{\perp}) = 0$.

By Lemma 2.7, $[h(ACQ)\text{-}h(A)h(CQ)]h(Q^{\perp}) = h(ACQQ^{\perp})\text{-}h(A)h(CQQ^{\perp}) = 0$.

An appeal to Lemma 2.5 gives $h(ACQ)\text{-}h(A)h(CQ) = 0$. $\square$

An ideal $\mathcal{I}$ of an algebra $\mathcal{A}$ is called a *separating set* of $\mathcal{A}$ if $\forall\, m \in \mathcal{A}$, $m\mathcal{I} = \{0\}$ implies $m = 0$ and $\forall\, n \in \mathcal{A}$, $\mathcal{I}n = \{0\}$ implies $n = 0$. The following simple result has many corollaries.

Theorem 2.8. *Suppose $\mathcal{I}$ is a separating set of a unital algebra $\mathcal{A}$ and $\mathcal{I}$ is contained in $\mathcal{R}$. If $h : \mathcal{A} \to \mathcal{B}$ is a bijective linear mapping from $\mathcal{A}$ to a unital algebra $\mathcal{B}$ satisfying $(*)$ and $h(I) = I$, then h is an isomorphism.*

Proof. First we claim that if $b_0 \in \mathcal{B}$ and $h(\mathcal{I})b_0 = \{0\}$, then $b_0 = 0$. Indeed, since h is surjective, we have $b_0 = h(a_0)$ for some $a_0 \in \mathcal{A}$. By (2.4), it follows that

$$h(\mathcal{I}a_0\mathcal{I}) = h(\mathcal{I})h(a_0)h(\mathcal{I}) = \{0\}.$$

Since h is injective, $\mathcal{I}a_0\mathcal{I} = \{0\}$. Since $\mathcal{I}$ is a separating set, $a_0 = 0$. Hence $b_0 = 0$.

Similarly, we can show that if $b_0 \in \mathcal{B}$ and $b_0 h(\mathcal{I}) = \{0\}$, then $b_0 = 0$.

By (2.3) and (2.4), $h(\mathcal{I})(h(xv)\text{-}h(x)h(v)) = \{0\}$ for every $x \in \mathcal{A}$ and $v \in \mathcal{I}$. Thus $h(xv) = h(x)h(v)$. Hence, for $x, y \in \mathcal{A}, v \in \mathcal{I}$, $h(xyv) = h(xy)h(v)$. On the other hand, $h(xyv) = h(x)h(yv) = h(x)h(y)h(v)$. Hence $(h(xy)\text{-}h(x)h(y))h(\mathcal{I}) = \{0\}$. Hence $h(xy) = h(x)h(y)$ for every $x, y \in \mathcal{A}$. $\square$

We do not know whether the conclusion of Theorem 2.1 remains valid without the assumption that h is bounded. The assumption is not needed for some special *CSL* algebras as indicated below by Corollaries 2.9 and 2.11.

A subspace lattice is called completely distributive if its subspaces satisfy the identity

$$\bigwedge_{i\in I}\bigvee_{j\in J} L_{ij} = \bigvee_{f\in J^I}\bigwedge_{i\in I} L_{if(i)},$$

where J^I denotes the set of all $f : I \to J$.

Corollary 2.9. *Let $\mathcal{L}$ be a completely distributive commutative subspace lattice on H. If h is a bijective linear mapping from $\operatorname{alg}\mathcal{L}$ onto a unital algebra $\mathcal{B}$ satisfying $(*)$ and $h(I) = I$, then h is an isomorphism.*

Proof. Let $\mathcal{I} = span\{T : T \in alg\,\mathcal{L},\ \operatorname{rank} T = 1\}$. Then $\mathcal{I}$ is an ideal of $alg\,\mathcal{L}$. From [8, Lemma 2.3], $\mathcal{I} \subseteq \mathcal{R}$. By [14, Theorem 3], it follows that $\mathcal{I}$ is a separating set of $alg\,\mathcal{L}$. By Theorem 2.8, h is an isomorphism. $\square$

Lemma 2.10 (*Pearcy and Topping [22]*). *Every operator in $B(H)$ is a sum of finite number of idempotents.*

Let $\mathcal{L}$ be a commutative subspace lattice. If $P, Q \in \mathcal{L}$ with $P \subsetneq Q$, then $Q\text{-}P$ is called an *interval*. Nests included in $\mathcal{L}$ are *independent* if the product of non-zero intervals, one taken from each nest, is again non-zero. Applying some techniques from [7], we can get the following corollary.

Corollary 2.11. *Let $\mathcal{L}$ be a commutative subspace lattice generated by finitely many independent nests $\mathcal{L}_1, \ldots, \mathcal{L}_n$ on H. If h is a bijective linear mapping from $\operatorname{alg}\mathcal{L}$ onto a unital algebra $\mathcal{B}$ satisfying $(*)$ and $h(I) = I$, then h is an isomorphism.*

Proof. Let $\Omega = \{i : I$ has no immediate predecessor in $\mathcal{L}_i\}$ and $\tilde{\Omega} = \{i : 0$ has no immediate successor in $\mathcal{L}_i\}$.

We divide the proof into four cases.

Case 1: Suppose that $\Omega = \emptyset$ and $\tilde{\Omega} = \emptyset$. Let $I_{i\text{-}}$ be the immediate predecessor of I in $\mathcal{L}_i$ and let 0_{i+} be the immediate successor of 0 in $\mathcal{L}_i$. Define $P = \prod_{i=1}^n 0_{i+}$, $Q = \prod_{i=1}^n (I_{i\text{-}})^\perp$ and

$$\mathcal{I} = span\{PA, BQ : A, B \in B(H)\}.$$

Case 2: Suppose that $\Omega \neq \emptyset$ and $\tilde{\Omega} \neq \emptyset$.

Let $P_1 = \prod_{i\notin\tilde{\Omega}} 0_{i+}$ and let $Q_1 = \prod_{i\notin\Omega}(I_{i\text{-}})^\perp$. If $i \in \tilde{\Omega}$, $t \in \Omega$ choose $P_{i,j} \neq 0, Q_{t,j} \neq I, P_{i,j} \in \mathcal{L}_i, Q_{t,j} \in \mathcal{L}_t, P_{i,j} \to 0$ and $Q_{t,j} \to I$ in strong operator topology as $j \to \infty$. Let $M_j = \prod_{i\in\tilde{\Omega}} P_{i,j}$ and $N_j = \prod_{i\in\Omega} Q_{i,j}$. Define

$$\mathcal{I} = span\{M_j P_1 A M_j^\perp, N_j B Q_1 N_j^\perp : A, B \in B(H), j = 1, 2 \ldots\}.$$

Case 3: Suppose that $\Omega \neq \emptyset$ and $\tilde{\Omega} = \emptyset$.

Define Q_1 and N_j the same as in Case 2. Let $P_1 = \prod_{i=1}^n 0_{i+}$.

Define

$$\mathcal{I} = span\{P_1 A, N_j B Q_1 N_j^\perp : A, B \in B(H), j = 1, 2 \ldots\}.$$

Case 4: Suppose that $\Omega = \emptyset$ and $\tilde{\Omega} \neq \emptyset$.

Define P_1 and M_j the same as in Case 2. Let $Q_1 = \prod_{i=1}^n (I_{i\text{-}})^\perp$.

Define

$$\mathcal{I} = span\{M_jP_1AM_j^{\perp}, BQ_1 : A, B \in B(H), j = 1, 2 \ldots\}.$$

If P is any projection in $\mathcal{L}$, by $PB(H) = PB(H)P^{\perp} + PB(H)P$ and Lemma 2.10, we have that $PB(H)$ is a linear span of its idempotents. Similarly, we can show that for any projection Q in $\mathcal{L}$, $B(H)Q$ is a linear span of its idempotents.

From Cases 1 to 4 and the previous paragraph, it can be verified that $\mathcal{I}$ is a separating set of $alg\,\mathcal{L}$. Now Theorem 2.8 implies h is an isomorphism. $\square$

In [13], Laurier gives an example of a commutative subspace lattice generated by two independent nests which is not completely distributive, thus Corollaries 2.9 and 2.11 are independent. Note that an isomorphism between a CLS algebra and any Banach algebra is automatically continuous, see [6]. For more on automatic continuity, we refer to [5].

Lemma 2.12 (*Longstaff [18]*). *Let $\mathcal{L}$ be a subspace lattice and $E \in \mathcal{L}$.*

(i) *If $e \in E$ and $f \in (E_-)^{\perp}$, then $e \otimes f \in alg\,\mathcal{L}$.*
(ii) *If $e \in E_+$ and $f \in E^{\perp}$, then $e \otimes f \in alg\,\mathcal{L}$.*

Corollary 2.13. *Let $\mathcal{L}$ be a subspace lattice with $0_+ \neq \{0\}$ and $X_- \neq X$. If h is a bijective linear mapping from $alg\,\mathcal{L}$ onto a unital algebra $\mathcal{B}$ satisfying $(*)$ and $h(I) = I$, then h is an isomorphism.*

Proof. Let $\mathcal{I} = span\{e \otimes f, g \otimes h : e \in 0_+, f \in X^*; g \in X, h \in X_-^{\perp}\}$. By Lemma 2.12, $\mathcal{I} \subseteq alg\,\mathcal{L}$. It is not hard to check that $\mathcal{I}$ is a separating set of $alg\,\mathcal{L}$.

Suppose $g \in X, h \in (X_-)^{\perp}$. If $h(g) \neq 0$, then $(h(g))^{-1}g \otimes h$ is an idempotent.

If $h(g) = 0$, choose $x \in X$ such that $h(x) = 1$, then $(g + x) \otimes h$ and $x \otimes h$ belong to $alg\,\mathcal{L}$ and are idempotents and $g \otimes h = (g + x) \otimes h - x \otimes h$.

Suppose that $e \in 0_+, f \in X^*$. Similarly, we can prove that $e \otimes f$ is a linear combination of some idempotents in $alg\,\mathcal{L}$.

By Theorem 2.8, we have that h is an isomorphism. $\square$

Applying Theorem 2.8 to $\mathcal{J}$-subspace lattices, one can have the following.

Corollary 2.14. *Let $\mathcal{L}$ be a $\mathcal{J}$-subspace lattice on X. If h is a bijective linear mapping from alg$\mathcal{L}$ onto a unital algebra $\mathcal{B}$ satisfying $(*)$ and $h(I) = I$, then h is an isomorphism.*

Proof. Let $\mathcal{I} = span\{T : T \in alg\,\mathcal{L},\ \text{rank}\ T = 1\}$. By [8,17], $\mathcal{I}$ is a separating set of $alg\,\mathcal{L}$ and every element of $\mathcal{I}$ can be expressed as a linear combination of idempotents in $alg\,\mathcal{L}$. Thus by Theorem 2.8, we have that h is an isomorphism. $\square$

Corollary 2.15. *Let $\mathcal{D}$ be a factor von Neumann algebra on H and let $\mathcal{N}$ be nest in $\mathcal{D}$. Suppose that h is a bijective linear mapping from $\mathcal{A} = (alg\,\mathcal{N}) \cap \mathcal{D}$ onto an unital algebra $\mathcal{B}$ satisfying $(*)$ and $h(I) = I$, then h is an isomorphism.*

Proof. By the proof of [9, Theorem 2.18], $\mathcal{A}$ has a separating set $\mathcal{I}$ and every element of $\mathcal{I}$ can be expressed as a linear combination of idempotents in $\mathcal{A}$. $\square$

3. Derivations and generalized derivations

It is well-known that if $\mathcal{A}$ is a 2-torsion free prime ring and δ is a Jordan derivation on $\mathcal{A}$, then δ is a derivation. In [12], Johnson gives a class of algebras which do not have non-trivial Jordan derivations. In this section, we study a class of non-selfadjoint algebras which do not have non-trivial Jordan derivations.

Suppose that $\mathcal{L}$ is a $\mathcal{J}$-subspace lattice on X. Denote $\mathcal{F}_{\mathcal{L}}$ the set of all finite-rank operators in $alg\,\mathcal{L}$. For $K \in \mathcal{J}_{\mathcal{L}}$, let $\mathcal{F}_K = span\{x \otimes f : x \in K, f \in K_-^{\perp}\}$.

Lemma 3.1. *Let $\mathcal{L}$ be a $\mathcal{J}$-subspace lattice on X and $\mathcal{M}$ be an $\mathcal{F}_{\mathcal{L}}$-bimodule. Suppose δ is a linear mapping from $\mathcal{F}_{\mathcal{L}}$ into $\mathcal{M}$ such that $\delta(P)=\delta(P)P+P\delta(P)$, for every idempotent P in $\mathcal{F}_{\mathcal{L}}$ and let K and M be two distinct elements in $\mathcal{J}_{\mathcal{L}}$. If $A=x\otimes f$ and $B=y\otimes g$, where $x\in K, f\in K_-^{\perp}, y\in M, g\in M_-^{\perp}$ are nonzero, then $AB=BA=0$ and $\delta(A)B+A\delta(B)=0$.*

Proof. Since $\mathcal{L}$ is a $\mathcal{J}$-subspace lattice, K and M belong to $\mathcal{J}_{\mathcal{L}}$, we have that $K\not\subseteq M$ and $M\not\subseteq K$. Thus $M\subseteq K_-$ and $K\subseteq M_-$, and it follows that $AB=BA=0$.

We divide the proof of the lemma into four cases.

Case 1: $f(x)=g(y)=1$. Then $A=A^2$ and $B^2=B$. Thus $(A+B)^2=A+B$. By the assumption, we have that

$$\delta(A)B=(\delta(A)A+A\delta(A))B=A\delta(A)B, A\delta(B)=A(\delta(B)B+B\delta(B))=A\delta(B)B,$$

$$\begin{aligned}\delta(A+B)&=\delta(A+B)(A+B)+(A+B)\delta(A+B)\\&=\delta(A)A+A\delta(A)+\delta(A)B+A\delta(B)+\delta(B)A+B\delta(A)+\delta(B)B+B\delta(B)\\&=\delta(A)+\delta(A)B+A\delta(B)+\delta(B)A+B\delta(A)+\delta(B).\end{aligned}$$

Thus

$$\delta(A)B+A\delta(B)+\delta(B)A+B\delta(A)=0. \tag{3.1}$$

By (3.1),

$$\delta(A)B+A\delta(B)=A\delta(A)B+A\delta(B)B=A[\delta(A)B+A\delta(B)+\delta(B)A+B\delta(A)]B=0.$$

Case 2: $f(x)=0, g(y)=1$. Since $K\vee K_-=X$ and $f\neq 0$, we can take $u\in K$ such that $f(u)=1$. Let $C=u\otimes f$. Then $A+C=(x+u)\otimes f$ and $f(x+u)=1$. By Case 1, we have

$$\delta(C)B+C\delta(B)=0, \delta(A+C)B+(A+C)\delta(B)=0.$$

Thus $\delta(A)B+A\delta(B)=0$.

Case 3: $f(x)=1, g(y)=0$. Similar to the proof in Case 2, we can show $\delta(A)B+A\delta(B)=0$.

Case 4: $f(x)=g(y)=0$. Choose w in M such that $g(w)=1$. Let $D=w\otimes g$. Then $B+D=(y+w)\otimes g$ and $g(y+w)=1$. By Case 2,

$$\delta(A)D+A\delta(D)=0, \delta(A)(B+D)+A\delta(B+D)=0.$$

Hence $\delta(A)B+A\delta(B)=0$. □

Lemma 3.2. *Let $\mathcal{A}=M_n(\mathbb{C})$ and $\mathcal{M}$ be an $\mathcal{A}$-bimodule. If $\delta:\mathcal{A}\to\mathcal{M}$ is a linear mapping such that $\delta(P)=\delta(P)P+P\delta(P)$ for every idempotent P in $\mathcal{A}$, then δ is a derivation.*

Proof. By [2, Lemma 1], it follows that δ is a Jordan derivation. By [12, Theorem 7.1], we have that δ is a derivation. □

Lemma 3.3. *Let $\mathcal{L}$ be a $\mathcal{J}$-subspace lattice on X and $\mathcal{M}$ be an $\mathcal{F}_{\mathcal{L}}$-bimodule. Suppose $\delta:\mathcal{F}_{\mathcal{L}}\to\mathcal{M}$ is a linear mapping such that $\delta(P)=\delta(P)P+P\delta(P)$ for every idempotent P in $\mathcal{F}_{\mathcal{L}}$, then δ is a derivation. In particular, every linear local Jordan derivation from $\mathcal{F}_{\mathcal{L}}$ to $\mathcal{M}$ is a derivation.*

Proof. For every $A,B\in\mathcal{F}_{\mathcal{L}}$, by [17, Theorem 3.2], we have that $A=A_1+\cdots+A_n$ and $B=B_1+\cdots+B_m$, where A_i and B_j are rank-one operator in $\mathcal{F}_{\mathcal{L}}$. By Lemma 2.12, there exist $K_i, M_j\in\mathcal{J}_{\mathcal{L}}$ such that $x_i\in K_i$, $f_i\in K_{i-}^{\perp}$, $y_j\in M_j$, $g_j\in M_{j-}^{\perp}$, $A_i=x_i\otimes f_i$, $B_j=y_j\otimes g_j$, $i=1,\ldots,n$ and $j=1,\ldots,m$.

By linearity of δ, to show $\delta(AB)=\delta(A)B+A\delta(B)$, we only need to show that for $K_i, M_j\in\mathcal{J}_{\mathcal{L}}$

$$\delta(A_iB_j)=\delta(A_i)B_j+A_i\delta(B_j).$$

Let $A=x\otimes f$ and $B=y\otimes g$ such that $x\in K, f\in K_-^{\perp}$ and $y\in M, g\in M_-^{\perp}$ are nonzero and $K,M\in\mathcal{J}_{\mathcal{L}}$.

Case 1: $M\neq K$. By Lemma 3.1, $AB=BA=0$ and $\delta(A)B+A\delta(B)=0$.

Case 2: $K=M$. Let $\mathcal{F}_K=span\{x\otimes f: x\in K, f\in K_-^{\perp}\}$. Then $\mathcal{F}_K$ is an ideal of $alg\,\mathcal{L}$. By [21, Lemma 3.6], we have that $\mathcal{F}_K$ is a locally matrix algebra.

Since $A,B\in\mathcal{F}_K$, by Lemma 3.2, we have that $\delta(AB)=\delta(A)B+A\delta(B)$. □

Corollary 3.4. *Let $\mathcal{M}$ be a right $\mathcal{F}_{\mathcal{L}}$-module. If $T : \mathcal{F}_{\mathcal{L}} \to \mathcal{M}$ is a linear mapping such that $T(P) = T(P)P$ for every idempotent P in $\mathcal{F}_{\mathcal{L}}$, then T is a left multiplier.*

Proof. For $A \in \mathcal{F}_{\mathcal{L}}$ and $m \in \mathcal{M}$, define $Am = 0$. Then $\mathcal{M}$ is an $\mathcal{F}_{\mathcal{L}}$-bimodule. By Lemma 3.3, we have that T is a left multiplier. □

The following lemma is essentially contained in [23].

Lemma 3.5. *Let $\mathcal{A}$ be a unital algebra and $\mathcal{M}$ be a unital $\mathcal{A}$-bimodule. If f is a left Jordan multiplier from $\mathcal{A}$ into $\mathcal{M}$, then f is a left multiplier.*

Proof. Since f is a left Jordan multiplier, we have that for every $x \in \mathcal{A}$, $f(x^2) = f(x)x$. So

$$f((x+y)^2) = (f(x+y))(x+y) = f(x)x + f(y)x + f(x)y + f(y)y \tag{3.2}$$

and

$$f((x+y)^2) = f(x^2 + xy + yx + y^2) = f(x)x + f(xy) + f(yx) + f(y^2). \tag{3.3}$$

By (3.2) and (3.3), it follows that

$$f(y)x = -(f(xy) - f(x)y).$$

So $f(x) - f(I)x = -(f(xI) - f(x)I) = 0$. Hence $f(x) = f(I)x$. For all $x, y \in \mathcal{A}$, $f(xy) = f(I)xy = f(x)y$. □

Theorem 3.6. *Let $\mathcal{L}$ be a $\mathcal{J}$-subspace lattice on X, $\mathcal{A}$ be the unital subalgebra of $alg\,\mathcal{L}$ generated by $\mathcal{F}_{\mathcal{L}}$, and $\mathcal{M}$ be a unital $\mathcal{A}$-bimodule. Then every linear generalized Jordan derivation from $\mathcal{A}$ to $\mathcal{M}$ is a generalized derivation.*

Proof. Since δ is a generalized Jordan derivation, it follows that $\delta(A^2) = \delta(A)A + A\tau(A)$, where τ is a Jordan derivation from $alg\,\mathcal{L}$ into itself. Let $T = \delta - \tau$. We have

$$T(A^2) = \delta(A^2) - \tau(A^2) = \delta(A)A + A\tau(A) - (\tau(A)A + A\tau(A)) = \delta(A)A - \tau(A)A = (\delta(A) - \tau(A))A = T(A)A$$

for every $A \in alg\,\mathcal{L}$.

So T is a left Jordan multiplier from $alg\,\mathcal{L}$ into $\mathcal{M}$. By Lemma 3.5, we have $T(AB) = T(A)B$, for all $A, B \in alg\,\mathcal{L}$. Thus

$$\delta(AB) - \tau(AB) = \delta(A)B - \tau(A)B. \tag{3.4}$$

Since τ is a Jordan derivation, it follows from Lemma 3.3 that τ is a derivation. By (3.4), it follows that

$$\delta(AB) = \delta(A)B + \tau(AB) - \tau(A)B = \delta(A)B + A\tau(B).$$

Hence δ is a generalized derivation. □

Remark 3.7. For any unital algebra $\mathcal{A}$ and a unital $\mathcal{A}$-bimodule $\mathcal{M}$, if all additive (linear) Jordan derivations from $\mathcal{A}$ to $\mathcal{M}$ are derivations then all additive (linear) generalized Jordan derivations from $\mathcal{A}$ to $\mathcal{M}$ are generalized derivations by Lemma 3.5 and the proof of Theorem 3.6.

We conclude this section by listing a few observations.

Since all additive Jordan derivations from a nest algebra on a Banach space X into $B(X)$ are derivations [15, Theorem 3.4], by Remark 3.7 we have the following corollary which generalizes the main result of [11].

Corollary 3.8. *If $\mathcal{N}$ is a nest on a Banach space X then every additive generalized Jordan derivation from $alg\,\mathcal{N}$ into $B(X)$ is a generalized derivation.*

Since all Jordan derivations from a *CSL* algebra to itself are derivations [20], by Remark 3.7 we have

Proposition 3.9. *If $\mathcal{L}$ is a commutative subspace lattice then every linear generalized Jordan derivation from $alg\,\mathcal{L}$ into itself is a generalized derivation.*

Lemma 3.10 (*Herstein [10]*). *Let δ be an additive Jordan derivation from an algebra $\mathcal{A}$ into an $\mathcal{A}$-bimodule. Then the following equalities hold.*

(1) $\delta(ab+ba) = \delta(a)b + a\delta(b) + \delta(b)a + b\delta(a)$,
(2) $\delta(aba) = \delta(a)ba + a\delta(b)a + ab\delta(a)$,
(3) $\delta(abc+cba) = \delta(a)bc + a\delta(b)c + ab\delta(c) + \delta(c)ba + c\delta(b)a + cb\delta(a)$ *for all* $a, b, c \in \mathcal{A}$.

Applying Lemma 3.10 together with some minor changes of the arguments used in the proofs of [15, Theorems 3.1 and 3.3], one can obtain the following.

Proposition 3.11. *Suppose $\mathcal{A}$ is a Banach subalgebra of $B(X)$ such that either $\mathcal{A}$ contains $\{x \otimes f : x \in X\}$, where $0 \neq f \in X^*$, or $\mathcal{A}$ contains $\{x \otimes f : f \in X^*\}$, where $0 \neq x \in X$. If $\delta : \mathcal{A} \to B(X)$ is a linear Jordan derivation, then δ is a derivation.*

References

[1] N. Argac, E. Albas, On generalized (σ, τ)-derivations, Siberian Math. J. 43 (2002) 977–984.
[2] M. Bresar, Characterization of derivations on some normed algebras with involution, J. Algebra 152 (1992) 454–462.
[3] M. Bresar, Jordan derivations revisited, Math. Proc. Cambridge Philos. Soc. 139 (2005) 411–425.
[4] M. Bresar, Characterizing homomorphisms, derivations and multipliers in rings with idempotents, Proc. R. Soc. Edinburgh Sect. A 137 (2007) 9–21.
[5] H. Dales, Banach Algebras and Automatic Continuity, Oxford Science Publications, 2000.
[6] K. Davidson, Nest Algebras, Pitman Research Notes in Mathematics, vol. 191, 1988.
[7] F. Gilfeather, A. Hopenwasser, D. Larson, Reflexive algebras with finite width lattices: tensor products, cohomology, compact perturbations, J. Funct. Anal. 55 (1984) 176–190.
[8] D. Hadwin, J. Li, Local derivations and local automorphisms, J. Math. Anal. Appl. 290 (2003) 702–714.
[9] D. Hadwin, J. Li, Local derivations and local automorphisms on some algebras, J. Operator Theory 60 (2008) 29–44.
[10] I.N. Herstein, Topics in Ring Theory, University of Chicago Press, Chicago, London, 1969.
[11] J. Hou, X. Qi, Generalized Jordan derivation on nest algebras, Linear Alg. Appl. 430 (2009) 1479–1485.
[12] B. Johnson, Symmetric amenability and the nonexistence of Lie and Jordan derivations, Math. Proc. Cambridge Philos. Soc. 120 (1996) 455–473.
[13] C. Laurier, On density of compact operators in reflexive algebras, Indiana Univ. Math. J. 30 (1981) 1–16.
[14] C. Laurie, W. Longstaff, A note on rank-one operators in reflexive algebras, Proc. Am. Math. Soc. 89 (1983) 293–297.
[15] J. Li, F. Lu, Additive Jordan derivations of reflexive algebras, J. Math. Anal. Appl. 329 (2007) 102–111.
[16] J. Li, Z. Pan, H. Xu, Characterizations of isomorphisms and derivations of some algebras, J. Math. Anal. Appl. 332 (2007) 1314–1322.
[17] J. Li, O. Panaia, Algebraic isomorphisms and $\mathcal{J}$-subspace lattices, Proc. Am. Math. Soc. 133 (2005) 2577–2587.
[18] W. Longstaff, Strongly reflexive lattices, J. London Math. Soc. 11 (1975) 491–498.
[19] W. Longstaff, O. Panaia, $\mathcal{J}$-subspace lattices and subspace $\mathcal{M}$-bases, Studia Math. 139 (2000) 197–212.
[20] F. Lu, The Jordan structure of CSL algebras, Studia Math. 190 (2009) 283–299.
[21] F. Lu, P. Li, Algebraic isomorphisms and Jordan derivations of $\mathcal{J}$-subspace lattice algebras, Studia Math. 158 (2003) 287–301.
[22] C. Pearcy, D. Topping, Sum of small numbers of idempotent, Michigan Math. J. 14 (1967) 453–465.
[23] B. Zalar, On centralizers of semiprime rings, Comment Math. Univ. Carolinae 32 (1991) 609–614.